My Research Results on Combinatorial Game Theory

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1 Introduction and the background of my research

In this article, I will summarize my research results for this year. In each section, I will write the introduction and to the theory that treated and present the main results of that study.

1.1 Combinatorial Game

My research is manily on combinatorial game theory. Combinatorial games are a class of two-player games with perfect information, where players take turns making moves, and the outcome of the game is determined solely by the positions and rules of the game. These games are characterized by the absence of chance or hidden information.

In this paper, let the name of two players be Left and Right.

1.2 Impartial Game and Partisan Game

Impartial games are games in which the sets of options for Left and Right are the same for each position. Impartial games have only two outcomes, $\mathcal{P}$-position and $\mathcal{N}$-position that are previous player's position and next player's position. partisan games are games in which the sets of options for Left and Right can be different. For the details of partisan games, see [1],

Most of my research is on impartial games, but I will also cover partisan games at the end of this article.

1.3 Nim

One of the most important topics of combinatorial game theory is Nim. There are few piles of stones. Each player takes their turns, and remove as many stones as she or he likes from one pile. The player who remove the last stones or a stone is the winner. The complete winning theory of classical theory was presented by C. Bouton in [2] for the first time in the history.

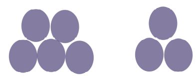

Figure 1.

1.4 Chocolate Game

The original chocolate games are presented in [3], and these games are mathematically the same as Nim. See Figures 2. A two-dimensional chocolate bar that I study is a rectangular array of squares in which some of the squares are removed. A bitter square printed in black is included in some parts of the bar. Figure3 displays an example of a two-dimensional chocolate bar. Each player takes their turn to break the bar in a straight line along the grooves and eats the broken piece without the bitter block. The player who manages to leave the opponent with the single bitter block (black block) is the winner.

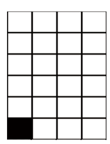

Figure 2: Step chocolate

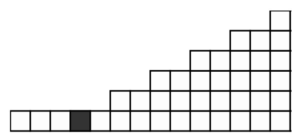

Figure 3: Three dimensional chocolate

1.5 Some background knowledge of combinatorial games

Here we quickly review some important concepts of combinatorial game theory. For the detail see [4] or [5].

Definition 1.1. *Let x and y be non-negative integers. They are expressed in Base 2 as follows: $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$, with $x_i, y_i \in \{0, 1\}$. We define the nim-sum $x \oplus y$ as follows:*

$$x \oplus y = \sum_{i=0}^{n} w_i 2^i,$$

where $w_i = x_i + y_i \pmod 2$.

For impartial games without drawings, there are only two outcome classes.

Definition 1.2. (i) *A position is referred to as a $\mathcal{P}$-position if it is a winning position for the previous player (the player who has just moved), as long as he plays correctly at every stage.*
(ii) *A position is referred to as an $\mathcal{N}$-position if it is a winning position for the next player as long as he plays correctly at every stage.*

Definition 1.3. (i) *For any position* $\mathbf{p}$ *of game* $\mathbf{G}$*, there is a set of positions that can be reached by precisely one move in* $\mathbf{G}$*, which we denote as move*$(\mathbf{p})$.
(ii) *The minimum excluded value (mex) of a set* S *of non-negative integers is the smallest non-negative integer that is not in* S.
(iii) *Let* $\mathbf{p}$ *be the position of an impartial game. The associated Grundy number is denoted as* $\mathcal{G}(\mathbf{p})$ *and is recursively defined as follows:* $\mathcal{G}(\mathbf{p}) = mex(\{\mathcal{G}(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\})$.

Definition 1.4. *The disjunctive sum of the two games, which is denoted as* $\mathbf{G} + \mathbf{H}$*, is a supergame in which a player may move in either* $\mathbf{G}$ *or* $\mathbf{H}$ *but not both.*

Theorem 1. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be impartial rulesets and* $\mathcal{G}_{\mathbf{G}}$ *and* $\mathcal{G}_{\mathbf{H}}$ *be the Grundy numbers of position* $\mathbf{g}$ *played under the rules of* $\mathbf{G}$ *and position* $\mathbf{h}$ *played under the rules of* $\mathbf{H}$*, respectively. Thus, we have the following:*
(i) *For any position* $\mathbf{g}$ *of* $\mathbf{G}$, $\mathcal{G}_{\mathbf{G}}(\mathbf{g}) = 0$ *if and only if* $\mathbf{g}$ *is a* $\mathcal{P}$ *position.*
(ii) *The Grundy number of positions* $\{\mathbf{g}, \mathbf{h}\}$ *in game* $\mathbf{G} + \mathbf{H}$ *is* $\mathcal{G}_{\mathbf{G}}(\mathbf{g}) \oplus \mathcal{G}_{\mathbf{H}}(\mathbf{h})$.

I will now present my research results on Combinatorial Game Theory.

2 Three-Dimensional Chocolate-Bar Games

A three-dimensional chocolate bar is a three-dimensional array of cubes in which a bitter cubic box printed in black is included in some part of the bar. Figure 4 displays an example of a three-dimensional chocolate bar. Each player takes their turn to cut the bar on a plane that is horizontal or vertical along the grooves, and eats the broken piece. The player who manages to leave the opponent with the single bitter cube is the winner. Examples of cut chocolate bars are depicted in Figures 5, 6, and 7.

Example 2.1. *Example of a three-dimensional chocolate bar.*

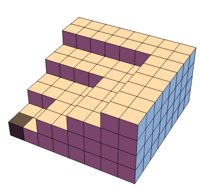

Figure 4. three dimensional chocolate

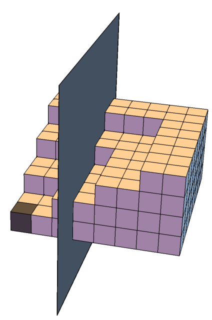

Figure 5. Vertical cut

Example 2.2.

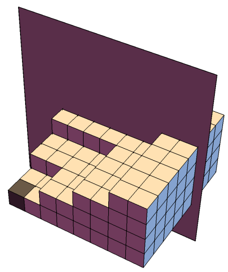

Figure 6. Vertical cut

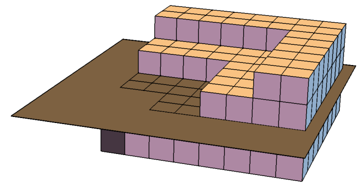

Figure 7. Horizontal cut

Definition 2.1. *Suppose* $f(u,v) \in Z_{\geq 0}$ *for* $u, v \in Z_{\geq 0}$. f *is said to monotonically increase if* $f(u,v) \leq f(x,z)$ *for* $x, z, u, v \in Z_{\geq 0}$ *with* $u \leq x$ *and* $v \leq z$.

We define a three-dimensional chocolate bar.

Definition 2.2. *Let* $x, y, z \in Z_{\geq 0}$ *such that* $y \leq f(x,y)$. *The three-dimensional chocolate bar comprises a set of* $1 \times 1 \times 1$ *boxes. For* $u, w \in Z_{\geq 0}$ *such that* $u \leq x$ *and* $w \leq z$, *the height of the column at position* (u,w) *is* $\min(f(u,w), y) + 1$, *where* f *is a monotonically increasing function in Definition 2.1. A bitter box exists at position* $(0,0)$. *We denote this chocolate bar by* $CB(f, x, y, z)$.
$x+1, y+1$, *and* $z+1$ *are the length, height, and width of the bar, respectively.*

We present a sufficient condition for a three-dimensional chocolate bar with length p, height q, and width r is P-position if and only if $(p-1) \oplus (q-1) \oplus (r-1) = 0$.

Theorem 2. *For a three-dimensional chocolate game whose three coordinates satisfy the inequality* $y \leq \lfloor \frac{x+z}{k} \rfloor$ *where* $k = 2^{a+1}(2m+1)$ *and* $x, z \leq (2^{2a+2} - 2^{a+1})m + 2^{2a+1} - 1$, *where* $a, m \in Z_{\geq 0}$ *The position (x,y,z) is a P-position If and only if* $x \oplus y \oplus z = 0$

This result has been already submitted to a journal, and it is now in the review process. See [6].

3 Restricted Nim with a Pass

Here I present my research result on the restricted Nim with a pass, and this result will be published soon as [7].

In this study, restricted Nim and restricted Nim with a pass are examined. An interesting but difficult question in combinatorial game theory has been to determine what happens when standard game rules are modified to allow a *one-time pass*, a pass move that may be used at most once in the game and not from a terminal position. Once either player has used a pass, it is no longer available. In the case of classical Nim, the introduction of the pass alters the mathematical structure of the game, considerably increasing its complexity. The effect of a pass on classical Nim remains an important open question that has defied traditional approaches. The late mathematician David Gale offered a monetary prize to the first person to develop a solution for three-pile classical Nim with a pass.

In [8] (p. 370), Friedman and Landsberg conjectured that "solvable combinatorial games are structurally unstable to perturbations, while generic, complex games will be structurally stable." One way to introduce such a perturbation is to allow a pass.

In the restricted Nim, the introduction of a pass move has a minimal impact. There is a simple relationship between the Grundy numbers of the game and the Grundy numbers of the game with a pass move, and the number of piles can be any natural number.

3.1 Maximum Nim

In this section, we study maximum Nim, which is a game of restricted Nim.

Definition 3.1. *If the sequence $f(m)$ for $m \in Z_{\geq 0}$ satisfies $0 \leq f(m) - f(m-1) \leq 1$ for any natural number m, it is called a regular sequence.*

Definition 3.2. *Let $f(m)$ be a regular sequence. Suppose there is a pile of n stones, and two players take turns removing stones from the pile. In each turn, the player is allowed to remove at least one stone and at most $f(m)$ stones, where m represents the number of stones. The player who removes the last stone is the winner. We refer to f as a rule sequence.*

Here, we let $f(x) = \lceil \frac{x}{2} \rceil$. Because $0 \leq f(m) - f(m-1) \leq 1$ for any $m \in N$, $f(m)$ for $m \in Z_{\geq 0}$ is a regular sequence. Here, we examine maximum Nim of Definition 3.2 for $f(x)$.

Definition 3.3. *We denote the pile of m stones as (m), which we call the position of the game.*

Definition 3.4. *The set of all the positions that can be reached from position (t) is defined as move(t). For any $t \in Z_{\geq 0}$, we have*

$$\textit{move}(t) = \{(t-v) : v \leq \left\lceil \frac{t}{2} \right\rceil \textit{ and } v \in N\}.$$

3.2 Three-Pile Maximum Nim

Definition 3.5. *Suppose that there are three piles of stones and two players take turns removing stones from the piles. In each turn, the player chooses a pile and removes at least one stone and at most $f(x) = \lceil \frac{x}{2} \rceil$ stones, where x*

represents the number of stones. The player who removes the last stone is the winner. The position of the game is represented by three coordinates (s, t, u), where s, t, and u represent the numbers of stones in the first, second, and third piles, respectively.

We can calculate the Grundy numbers of the game in Definition 3.5.

Theorem 3. *Let $\mathcal{G}(t)$ be the Grundy number of the game in Subsection 3.1. Then, the Grundy number $\mathcal{G}(s, t, u)$ of the game of Definition 3.5 satisfies the following equation: $\mathcal{G}(s, t, u) = \mathcal{G}(s) \oplus \mathcal{G}(t) \oplus \mathcal{G}(u)$.*

Proof. This is directly from Theorem 1. $\square$

3.3 Maximum Nim with a Pass

In Subsections 3.4 and 3.5, we modify the standard rules of the games to allow for a one-time pass, that is, a pass move that may be used at most once in the game and not from a terminal position. Once a pass has been used by either player, it is no longer available.

3.4 Maximum Nim with a Pass Whose Rule Sequence Is $f(x) = \lceil \frac{x}{2} \rceil$

The position of this game is represented by two coordinates (t, p), where t represents the number of stones in the pile. We define $p = 1$ if the pass is still available; otherwise, $p = 0$.

We define *move* in this game.

Definition 3.6. *For any $t \in Z_{\geq 0}$, we have (i) and (ii).*
(i) If $p = 1$ and $t > 0$,

$$move(t, p) = \{(t - v, p) : v \leq \left\lceil \frac{t}{2} \right\rceil \text{ and } v \in N\} \cup \{(t, 0)\}.$$

(ii) If $p = 0$ or $t = 0$,

$$move(t, p) = \{(t - v, p) : u \leq \left\lceil \frac{t}{2} \right\rceil \text{ and } v \in N\}.$$

Remark 3.1. *Note that a pass is unavailable from position $(t, 1)$ with $t = 0$, which is the terminal position. Apparently, $\mathcal{G}(t, 0)$ is identical to $\mathcal{G}(t)$ in Section 3.1 .*

According to Definitions 1.3 and 3.6, we define the Grundy number $\mathcal{G}(t, p)$ of the position (t, p).

p \ t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	0	1	0	2	1	3	0	4	2	5	1	6	3	7	0	8	4	9	2	10	5	11	1	12	6	13
1	0	2	1	0	2	4	1	3	0	6	2	5	4	8	1	7	3	10	0	9	6	12	2	11	5	14

Figure 8: Table of Grundy numbers $\mathcal{G}(t, p)$

The following Theorem is results of my research.

Theorem 4. *Let $\mathcal{G}(s,p)$ be the Grundy number of position (s,p). Then, we obtain the following:*
(i) $\mathcal{G}(0,0)=0$ *and* $\mathcal{G}(0,1)=0$.
(ii) For $u \in N$, *if* $\mathcal{G}(u,0)=0$, *then* $\mathcal{G}(u,1)=1$.
(iii) For $u \in N$, *if* $\mathcal{G}(u,0)=2$, *then* $\mathcal{G}(u,1)=0$.
(iv) For $u,m \in N$ *such that* $m>1$, *if* $\mathcal{G}(u,0)=2m$, *then* $\mathcal{G}(u,1)=2m-1$.
(v) For $u,m \in N$, *if* $\mathcal{G}(u,0)=2m-1$, *then* $\mathcal{G}(u,1)=2m$.

3.5 Three-Pile Maximum Nim with a Pass

Here, we study maximum Nim with three piles based on Definition 3.5 by modifying the standard rules of the games to allow a one-time pass. We consider only three-pile games, although generalization to the case of an arbitrary natural number of piles is straightforward.

We denote the position of the game with four coordinates (s,t,u,p), where s, t, and u represent the numbers of stones in the first, second, and third piles, respectively. Here, we define $p=1$ if the pass is still available, and $p=0$ otherwise.

We define a *move* in this game as follows.

Definition 3.7. *For any* $s,t,u \in Z_{\geq 0}$, *we have (i) and (ii).*
(i) If $p=1$ *and* $s+t+u>0$,

$$\begin{aligned} move(s,t,u,p) = {} & \{(s-v,t,u,p) : v \leq \left\lceil \frac{s}{2} \right\rceil \text{ and } v \in N\} \\ & \cup \{(s,t-v,u,p) : v \leq \left\lceil \frac{t}{2} \right\rceil \text{ and } v \in N\} \\ & \cup \{(s,t,u-v,p) : v \leq \left\lceil \frac{u}{2} \right\rceil \text{ and } v \in N\} \cup \{(s,t,u,0)\}. \end{aligned}$$

(ii) If $p=0$ *or* $s+t+u=0$,

$$\begin{aligned} move(s,t,u,p) = {} & \{(s-v,t,u,p) : v \leq \left\lceil \frac{s}{2} \right\rceil \text{ and } v \in N\} \\ & \cup \{(s,t-v,u,p) : v \leq \left\lceil \frac{t}{2} \right\rceil \text{ and } v \in N\} \\ & \cup \{(s,t,u-v,p) : v \leq \left\lceil \frac{u}{2} \right\rceil \text{ and } v \in N\}. \end{aligned}$$

According to Definitions 1.3 and 3.7, we define the Grundy number $\mathcal{G}(s,t,u,p)$ of the position (s,t,u,p). Because s, t, and u represent the numbers of stones in the first, second, and third piles, respectively, the value of $\mathcal{G}(s,t,u,p)$ does not depend on the order of the arguments s,t,u.

Remark 3.2. *Note that a pass is unavailable from the position* $(s,t,u,1)$ *with* $s+t+u=0$, *which is the terminal position. It is clear that* $\mathcal{G}(s,0,0,p)$, $\mathcal{G}(0,s,0,p)$, *and* $\mathcal{G}(0,0,s,p)$ *are identical to* $\mathcal{G}(s,p)$ *in Section 3.4.*

This is my main result of the research of [7].

Theorem 5. *We have the following for the Grundy numbers:*
(i) *For* $s \in Z_{\geq 0}$, $\mathcal{G}(s,0,0,1) = \mathcal{G}(0,s,0,1) = \mathcal{G}(0,0,s,1) = \mathcal{G}(s,1)$.
(ii) *We suppose that* $s,t > 0$, $t,u > 0$, *or* $u,s > 0$. *Thus, we have the following:*
$(ii.1)$ *For any* $m \in Z_{\geq 0}$, *if* $\mathcal{G}(s,t,u,0) = 2m$, *then* $\mathcal{G}(s,t,u,1) = 2m+1$.
$(ii.2)$ *For any* $m \in Z_{\geq 0}$, *if* $\mathcal{G}(s,t,u,0) = 2m+1$, *then* $\mathcal{G}(s,t,u,1) = 2m$.

As easily seen, Theorem 5 gives simple.

4 Symbolic Regression to Unsolved Mathematical Problems

The result of this research is presented in [9]. This study proposes a method to solve unsolved mathematical games using the symbolic regression library. In mathematics, mathematicians spend time finding a formula that describes a given data, which is often time-consuming. Here symbolic regression library can help humans.

First, I with the help of my teachers and coaches customized a Python symbolic regression library "gplearn" by implementing new features, such as conditional branching, and a few discrete functions selected for the study. "gplearn" uses generic programming to find a formula from data. We found that the performance of customized "gplearn" is far better than the original one. In this customized "gplearn", the person who used this library should set the condition of branching, and we used their knowledge as mathematicians when they set the conditions.

Secondly, they made a Swift symbolic regression library using generic programming. In this library, we implemented a new method to select the fittest formulas by combining two methods of choosing formulas. The first method is to choose by the smallness of the mean absolute error, and the second method is to choose by the largeness of the percentage of the given data that can be described without any error by a specific formula. As a result, the Swift library can discover formulas as good as customized gplearn without using conditional branching. Therefore, this Swift library can perform without the knowledge of specialists in the field of research. In some examples, the Swift library could discover fewer formulas that describe the data than the customized gelearn. Therefore, it is undoubtedly a better library for a specific field of mathematics.

The research result shows the possibility of using generic programming in mathematics and widens the scope of the research on symbolic regression.

5 Partisan Chocolate Games

In this study, we study Partisan Chocolate Games. This is my joint research with Dr.T. Abuku, Prof. R. Nowakowski, Prof. C. P. Santos and Dr. K. Suetsugu. So far, impartial chocolate games are studied as a generalization of

Nim. In this study, we use color blocks of chocolates and restrict the player who can make a move to one player for each line. In general, under this rule, there are various values. However, in the checkerboard pattern and stripe pattern, the values are only numbers, and we found formulas to determine the values.

Definition 5.1. *In the partisan game, there are two other outcome classes beside $\mathcal{P}$-position and $\mathcal{N}$-position in Section 1.2. We need the following outcome classes.*
(a) A position is referred to as a $\mathcal{L}$-position if it is a winning position for Left.
(b) A position is referred to as an $\mathcal{R}$-position if it is a winning position for Right.

5.1 Game values

In partisan game, the study of game values is very important for analyzing game's outcome classes. In the following, we briefly discuss the values of the games, but omit the description of games with special values.

Let G be a position of the game. Let G^L denote the left options of G (the positions after a left move in G), and let G^R denote the right options of G (the positions after a right move in G) as well. The game G will be denoted by the following.

$$G = \{G^L \mid G^R\}$$

In the end game, no options exist. Such a game is called an empty game and is written as $\{\emptyset \mid \emptyset\}$ or $\{\ \mid\ \}$.

Definition 5.2. *For $n \in \mathbb{Z}$, game value is defined as follows:*
(i)$0 \overset{\text{def}}{=} \{\emptyset \mid \emptyset\}$,
(ii)$n \overset{\text{def}}{=} \{n-1 \mid\ \}$,
(iii) $-n \overset{\text{def}}{=} \{\ \mid -(n-1)\}$.
For binary rational numbers, we define the following: For $j > 0$ and odd m,
(iv)$\dfrac{m}{2^j} \overset{\text{def}}{=} \left\{ \dfrac{m-1}{2^j} \middle| \dfrac{m+1}{2^j} \right\}$.

5.2 Hackenbush

Hackenbush is a game played on a graph. Every edge is colored blue or red. There is a special vertex called "ground" and other vertices. In her turn, Left (resp. Right) chooses one bLue (resp. Red) edge and removes it. If deleting an edge splits the original graph into two disconnected components, the connected component that does not contain "ground" is deleted at the same time.

5.3 Rules of the Partisan Chocolate Game

Let $\mathbb{N}_0$ be the set of all non-negative integers. We consider the chocolate game to be the partisan version, it is called a Partisan Chocolate Game.

We define the Partisan Chocolate Game.

Definition 5.3. *In Partisan Chocolate Game, except for the leftmost-bottom bitter block, every block is colored blue or red.*

For each column of blocks, if the top block is colored blue, Left can cut the chocolate in two along the vertical line to its left. If the top block is colored red, Right can cut the chocolate in two along the vertical line to its left. For each row of blocks, if the rightmost block is colored blue, Left can cut the chocolate in two along the horizontal line to its bottom. If the rightmost block is colored red, Right can cut the chocolate in two along the horizontal line to its bottom.

After cutting, the player eats the piece which does not include the bitter block and the player who cannot cut the board is the loser as well as the original Chocolate Game.

Note that the player can cut the board from top or right, so in positions like Figure 9, Left does not have any legal moves.

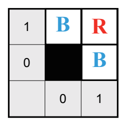

Figure 9: Position in which Left has no legal move

Example 6. *We have a checkerboard pattern as the one in Figure 10. Left can cut the board as in Figure 12. Similarly, Right can cut the board as in Figure 13.*

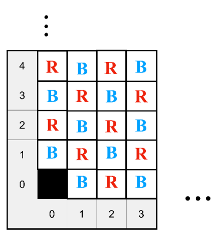

Figure 10: Checkerboard pattern

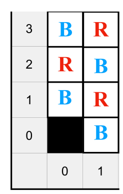

Figure 11: Position $(3, 1)$

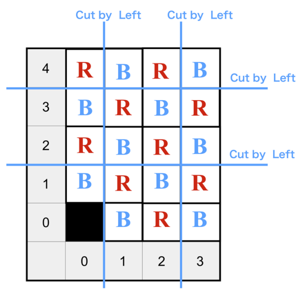

Figure 12: Cut by Left

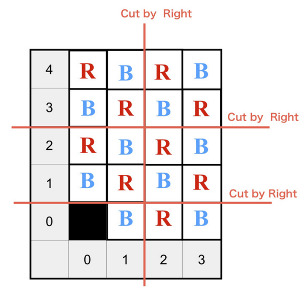

Figure 13: Cut by Right

The position or the shape of $n \times m$ chocolates is denoted by $(n-1, m-1)$, so the positions of chocolates in Figures 11 and 12 are described by $(1, 3)$ and $(3, 4)$.

5.4 In the case of $1 \times n$

Every position in $1 \times n$ Partisan Chocolate Game is the same as a position Hackenbush string with corresponding blue and red edges. See the following example.

Example 7. *The position in Figure 14 is isomorphic to the position in Figure 15.*

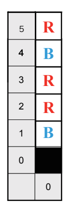

Figure 14: One-dimensional chocolate

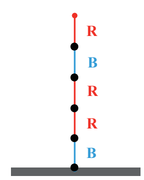

Figure 15: Hackenbush

One can calculate the value of any position in Hackenbush string by using a simple method. Thus, one can calculate the value of any position in $1 \times n$ Partisan Chocolate Game. For the details of Hackenbush string, see [5].

Definition 5.4. *We define the sequence* $\{U_n : n \in \mathbb{N}_0\}$ *by*

$$\begin{cases} U_0 = 0, \\ U_n = \displaystyle\sum_{i=0}^{n-1} \left(-\frac{1}{2}\right)^i & \textit{if } \ n > 0. \end{cases}$$

We can calculate the value of any position in the checkerboard pattern by following Theorem .

Theorem 8. *For any $x, y \in \mathbb{N}_0$, we have the following equations:*
(i) $\mathrm{V}(2x, 2y) = U_{2x+2y}$,
(ii) $\mathrm{V}(2x+1, 2y) = \mathrm{V}(2x, 2y+1) = U_{2x+2y+1}$,
(iii) $\mathrm{V}(2x+1, 2y+1) = U_{2x+2y}$.

For a proof, see [10].

5.5 The case of $n \times m$ stripeboard pattern.

Next, we consider stripeboard pattern. The blocks of the leftmost column are red except for the bottom bitter block, the blocks of the second column are blue, the blocks of the third column are red, and continue alternately. See Figure 16.

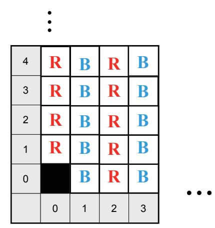

Figure 16: Stripe-board pattern

We can calculate the value of any position in the stripeboard pattern by following Theorem .

Theorem 9. *For any $x, y \in \mathbb{N}_0$, we have the following equations:*

$$V(x,y) = \begin{cases} U_{x-2y} & (x \geq 2y), \\ \dfrac{x-2y}{2} & (x < 2y \quad \textit{and} \quad x \quad \textit{is even}), \\ \dfrac{-x+2y+3}{2} & (x < 2y \quad \textit{and} \quad x \quad \textit{is odd}). \end{cases}$$

For a proof, see [10].

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Appendix A: Publications

(1) R. Miyadera and H. Manabe, Restricted Nim with a Pass. To appear in Integers.

(2) R. Miyadera, S. Nakamura and H. Manabe, Previous Player's Positions of Impartial Three-Dimensional Chocolate-Bar Games. To appear in Thai Journal of Mathematics.

(3) K. Tanemura, Y. Tachibana, Y. Tokuni, H. Manabe and R. Miyadera, Application of Generic Programming to Unsolved Mathematical Problems, Proceeding of IEEE 11th Global Conference on Consumer Electronics (GCCE), Osaka, Japan, 2022, pp. 845-849

(4) T. Abuku, H. Manabe, R. J. Nowakowski, C. P. Santos, Koki Suetsugu "Partisan Chocolate Game", IPSJ SIG Technical Report, Vol. 2023-GI-49 No.7, pp. 1-7, 2023

Appendix B: Conferences

(1) Y. Tokuni, R. Miyadera, H. Manabe, A. Murakami and S. Takahasi. "Arithmetic Mean and Geometric Mean"The international conference "The 24th Japan Conference on Discrete and Computational Geometry, Graphs, and Games "September 9-11, 2022 at Tokyo University of Science, Tokyo, Japan.

(2) K. Tanemura, R. Miyadera, Y. Tokuni and H. Manabe. "Combinatorial Games and Genetic Programming" The international conference "The 24th Japan Conference on Discrete and Computational Geometry, Graphs, and Games" September 9-11, 2022 at Tokyo University of Science, Tokyo, Japan.

(3) R. Miyadera, H. Manabe and A. Singh. "P-positions of Chocolate Games" The international conference "The 24th Japan Conference on Discrete and Computational Geometry, Graphs, and Games" September 9-11, 2022 at Tokyo University of Science, Tokyo, Japan.

(4) R. Miyadera, H. Manabe and A. Singh. "Previous Player's Positions of Three-Dimensional Chocolate Games with a Restriction on the Size of Chocolate The international conference "The 24th Japan Conference on Discrete and Computational Geometry, Graphs, and Games" September 9-11, 2022 at Tokyo University of Science, Tokyo, Japan.

(5) K. Tanemura, Y. Tachibana, Y. Tokuni, H. Manabe and R. Miyadera, Application of Generic Programming to Unsolved Mathematical Problems 2022 IEEE 11th Global Conference on Consumer Electronics (GCCE 2022).

(6) Y. Sasaki, K. Tanemura, Y. Tokuni, R. Miyadera and H. Manabe, Application of Symbolic Regression to Unsolved Mathematical Problems, Alliance Technology Conference-1-International Conference on Artificial Intelligence and Applications-2023 (ICAIA), Bengaluru, India.

Appendix C: Awards

(1) "49th Information Processing Society of Japan" Young Researcher Award, https://www.ipsj.or.jp/award/gi-award1.html

(2) Kadokawa dwango Educational institute, Academic Research Club Special Research Presentation Grand Prize